

A COHOMOLOGICAL PAIRING OF HALF-FORMS

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ABSTRACT. Blattner and Rawnsley have constructed half-forms for regular polarizations of arbitrary index. We show how to pair these half-forms into a line bundle fashioned purely from the symplectic data, with no assumption on the intersection of the polarizations. Our pairing agrees with the regular BKS pairing when the polarizations are positive.

Introduction. Half-forms and their pairings arise in geometric quantization, where they serve in the construction and comparison of representations of Poisson algebras of functions on a symplectic manifold. Half-forms were first defined for positive polarizations [3]. In this context, half-form pairings were presented initially for transverse polarizations [3] and later for polarizations with regular intersection [1, 8]. In [2] it is shown how to define half-forms for arbitrary polarizations. Our purpose here is to exhibit half-form pairings in this general setting. We shall develop the implications for geometric quantization in a later article.

Fundamental to our pairing is the consideration of a symplectic construction and its relationship with the metaplectic representation.

Let (V, Ω) be a symplectic vector space; equip the direct sum $V_{\#} = V \oplus V$ with the symplectic form $\Omega_{\#}$ given by $+\Omega$ on the first factor and $-\Omega$ on the second. Let W be an infinite-dimensional irreducible unitary projective representation of V on a Hilbert space $\mathbf{H}$, with multiplier $\frac{1}{i\hbar}\Omega$ for some positive real number $\hbar$. W determines a (two-sided) irreducible unitary projective representation $W_{\#}$ of $V_{\#}$ on the Hilbert space $\mathbf{H}_{\#}$ of Hilbert-Schmidt operators on $\mathbf{H}$, with multiplier $\frac{1}{i\hbar}\Omega_{\#}$. Conjugation also defines a map from the group $\text{Mp}^c(V, \Omega)$ of automorphisms of W to the group $\text{Mp}^c(V_{\#}, \Omega_{\#})$ of automorphisms of $W_{\#}$; moreover, this map factors through the symplectic group $\text{Sp}(V, \Omega)$ and takes values in the metaplectic group $\text{Mp}(V_{\#}, \Omega_{\#})$, to define a map $\theta: \text{Sp}(V, \Omega) \rightarrow \text{Mp}(V_{\#}, \Omega_{\#})$.

Let F and G be arbitrary polarizations of (V, Ω) ; the external direct sum $F \oplus \overline{G} \subset V_{\#}^{\mathbb{C}}$ is then a polarization of $(V_{\#}, \Omega_{\#})$. The vacuum states for F and G are one-dimensional cohomology spaces for F and G , considered as abelian Lie algebras, with coefficients in the symplectic spinors $\mathcal{E}'$ defined by W ; the vacuum state for $F \oplus \overline{G}$ is fashioned likewise from cohomology with coefficients in the symplectic spinors $\mathcal{E}'_{\#}$ defined by $W_{\#}$. A Künneth theorem for this cohomology provides a pairing of the vacuum states for F and G into the vacuum state for $F \oplus \overline{G}$.

Let (E, ω) be a symplectic vector bundle; fiberwise application of $\#$ produces a symplectic vector bundle $(E_{\#}, \omega_{\#})$. The map θ endows $(E_{\#}, \omega_{\#})$ with a canonical metaplectic structure; this allows us to globalize the pairing of vacuum states and

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define a half-form pairing for arbitrary polarizations of (E, ω) . For polarizations F and G of (E, ω) , our pairing takes its values in the bundle of half-forms for the polarization $F \oplus \overline{G}$ of $(E_{\#}, \omega_{\#})$.

Some elementary algebra of symplectic vector spaces is presented in §1; this material is quite standard. A brief account of the metaplectic representation and symplectic spinors is contained in §2; details may be found in [7]. In §3 we recall the definition of vacuum states for arbitrary polarizations and pair them using techniques from the cohomology of Lie algebras. Mp^c and metaplectic structures form the subject matter of §4: we discuss their existence and classification and recall how they give rise to bundles of half-forms. Finally, the constructions of §§3 and 4 are combined in §5 to define our pairing of half-forms for arbitrary polarizations.

1. Symplectic algebra. Let (V, Ω) be a real symplectic vector space: a real vector space V equipped with a nonsingular alternating real bilinear form Ω . We assume throughout that V is of dimension $2m$. The symplectic group $\text{Sp}(V, \Omega)$ consists of all linear automorphisms g of V satisfying $\Omega(gx, gy) = \Omega(x, y)$ whenever $x, y \in V$.

Complexification of V and complex-bilinear extension of Ω give rise to a complex symplectic vector space $(V^{\mathbb{C}}, \Omega^{\mathbb{C}})$. A Hermitian form (of zero signature) is defined on $V^{\mathbb{C}}$ by $H(x, y) = i\Omega^{\mathbb{C}}(x, \overline{y})$ for $x, y \in V^{\mathbb{C}}$; here, an upper bar denotes conjugation in $V^{\mathbb{C}}$ relative to V .

A polarization of (V, Ω) is a Lagrangian in $(V^{\mathbb{C}}, \Omega^{\mathbb{C}})$: thus, a complex subspace F of $V^{\mathbb{C}}$ such that $\dim_{\mathbb{C}} F = m$ and

$$x, y \in F \implies \Omega^{\mathbb{C}}(x, y) = 0.$$

The real part $F \cap \overline{F}$ of F is the null space of H on F ; we write $r(F) = \dim_{\mathbb{C}}(F \cap \overline{F})$. The complex dimension of a maximal subspace of F on which H is positive-definite (respectively, negative-definite) will be denoted $p(F)$ (respectively, $q(F)$). We refer to $(p(F), q(F), r(F))$ as the type of F and say that F is

$$\begin{array}{ll} \text{real} & \text{iff } 0 = p(F) = q(F); \\ \text{positive} & \text{iff } 0 = q(F); \\ \text{strictly positive} & \text{iff } 0 = r(F) = q(F). \end{array}$$

The symplectic group $\text{Sp}(V, \Omega)$ acts on the space $\text{Lag}(V^{\mathbb{C}}, \Omega^{\mathbb{C}})$ of all polarizations of (V, Ω) as follows: if $g \in \text{Sp}(V, \Omega)$ and $F \in \text{Lag}(V^{\mathbb{C}}, \Omega^{\mathbb{C}})$ then

$$g \cdot F = \{g^{\mathbb{C}}(v) \mid v \in F\}.$$

Orbits for this action are characterized by type [5, 10]: if F and G are polarizations of (V, Ω) then F and G lie in the same orbit under $\text{Sp}(V, \Omega)$ iff

$$(p(F), q(F), r(F)) = (p(G), q(G), r(G)).$$

On the direct sum $V_{\#} = V \oplus V$ we define a nonsingular alternating bilinear form $\Omega_{\#}$ by

$$\Omega_{\#}(x_1 \oplus y_1, x_2 \oplus y_2) = \Omega(x_1, x_2) - \Omega(y_1, y_2)$$

for $x_1, x_2, y_1, y_2 \in V$. In this way we produce a symplectic vector space $(V_{\#}, \Omega_{\#})$ of which the diagonal

$$V_{\Delta} = \{v \oplus v \mid v \in V\}$$

is a distinguished Lagrangian. A natural morphism

$$\phi : \mathrm{Sp}(V, \Omega) \rightarrow \mathrm{Sp}(V_{\#}, \Omega_{\#}) : g \mapsto g_{\#}$$

of Lie groups is defined by

$$g_{\#}(x \oplus y) = (gx) \oplus (gy)$$

for $x, y \in V$.

Let F and G be polarizations of (V, Ω) . The (external) direct sum

$$F \oplus \overline{G} \subset V^{\mathbf{C}} \oplus V^{\mathbf{C}} = V_{\#}^{\mathbf{C}}$$

is then a polarization of $(V_{\#}, \Omega_{\#})$ whose type is the sum of those for F and G ; in particular, $q(F \oplus \overline{G}) = q(F) + q(G)$.

2. The metaplectic representation. Let W be an irreducible unitary projective representation of the additive group of V on a Hilbert space $\mathbf{H}$; let W have multiplier $\frac{1}{i\hbar}\Omega$ where $\hbar = 2\pi\hbar$ is a positive real number. If $x, y \in V$ then

$$W(x)W(y) = \exp \left\{ \frac{1}{2i\hbar}\Omega(x, y) \right\} W(x + y).$$

Let $g \in \mathrm{Sp}(V, \Omega)$ and define $W_g(v) = W(gv)$ for $v \in V$; W_g is then also an irreducible unitary projective representation of V with multiplier $\frac{1}{i\hbar}\Omega$. According to the uniqueness theorem of Stone and von Neumann, W_g is unitarily equivalent to W : there exists a unitary operator U on $\mathbf{H}$ such that

$$(2.1) \quad v \in V \Rightarrow UW(v)U^{-1} = W(gv).$$

Since W is irreducible, U is unique up to scalar multiples; moreover U uniquely determines g .

We denote by $\mathrm{Mp}^c(V, \Omega)$ the subgroup of the unitary group $\mathrm{Aut} \mathbf{H}$ consisting of all unitary operators U on $\mathbf{H}$ satisfying (2.1) for some $g \in \mathrm{Sp}(V, \Omega)$ and write $\sigma(U) = g$ when (2.1) holds. $\mathrm{Mp}^c(V, \Omega)$ is a Lie group and

$$1 \rightarrow U(1) \hookrightarrow \mathrm{Mp}^c(V, \Omega) \xrightarrow{\sigma} \mathrm{Sp}(V, \Omega) \rightarrow 1$$

is a central short exact sequence. See [8 and 9].

The group of unitary characters on $\mathrm{Mp}^c(V, \Omega)$ is infinite cyclic; as generator we may take the unique unitary character $\eta : \mathrm{Mp}^c(V, \Omega) \rightarrow U(1)$ that restricts to $U(1) \subset \mathrm{Mp}^c(V, \Omega)$ as the squaring map. The kernel of η is a connected double cover of $\mathrm{Sp}(V, \Omega)$ and so deserves to be called the metaplectic group $\mathrm{Mp}(V, \Omega)$. Inclusion of $\mathrm{Mp}^c(V, \Omega)$ in $\mathrm{Aut} \mathbf{H}$ defines a faithful unitary representation of $\mathrm{Mp}^c(V, \Omega)$ on $\mathbf{H}$; this is known variously as the harmonic, oscillator, Segal-Shale-Weil, or metaplectic representation.

W differentiates on its dense space $\mathcal{E} \subset \mathbf{H}$ of smooth vectors, to give $\dot{W} : V \rightarrow \mathrm{End} \mathcal{E}$ satisfying

$$(2.2) \quad [\dot{W}(x), \dot{W}(y)] = \frac{1}{i\hbar}\Omega(x, y)I$$

for $x, y \in V$; $i\hbar\dot{W}$ therefore gives a representation of the canonical commutation relations. $\mathcal{E}$ is naturally provided with the structure of a Fréchet space; we equip the conjugate-linear dual $\mathcal{E}'$ with the weak-star topology and refer to its elements

as symplectic spinors. The Hilbert structure embeds $\mathbf{H}$ in $\mathcal{E}'$, and $\mathcal{E} \subset \mathbf{H} \subset \mathcal{E}'$ is a rigged Hilbert space in the sense of Gelfand. $\dot{W}$ extends continuously to $\mathcal{E}'$ and then complexifies to yield

$$(2.3) \quad \dot{W}^{\mathbf{C}}: V^{\mathbf{C}} \rightarrow \text{End } \mathcal{E}'.$$

See [3 and 8].

Any unitary operator on $\mathbf{H}$ stabilizing $\mathcal{E}$ extends to an automorphism of $\mathcal{E}'$; the metaplectic representation of $\text{Mp}^c(V, \Omega)$ therefore extends to $\mathcal{E}'$. If $U \in \text{Mp}^c(V, \Omega)$ with $\sigma(U) = g$ and $v \in V^{\mathbf{C}}$ then

$$(2.4) \quad U\dot{W}^{\mathbf{C}}(v)U^{-1} = \dot{W}^{\mathbf{C}}(gv),$$

as follows by differentiation, extension, and complexification of (2.1).

Let $\mathbf{H}_{\#}$ denote the Hilbert space of all Hilbert-Schmidt operators on $\mathbf{H}$; recall that the linear endomorphism T of $\mathbf{H}$ is Hilbert-Schmidt iff the series $\sum_{i \in I} \|Te_i\|^2$ converges for $(e_i)_{i \in I}$ a complete orthonormal system in $\mathbf{H}$.

For $T \in \mathbf{H}_{\#}$ and $x, y \in V$ we define

$$W_{\#}(x \oplus y)T = W(x)TW(y)^{-1}.$$

(2.5) THEOREM. $W_{\#}$ is an irreducible unitary projective representation of $V_{\#}$ with multiplier $\frac{1}{i\hbar}\Omega_{\#}$.

PROOF. That $W_{\#}$ is irreducible and unitary is standard; that $W_{\#}$ has the stated multiplier is a matter of routine verification. $\square$

In terms of $W_{\#}$ we define the central circle extension $\text{Mp}^c(V_{\#}, \Omega_{\#})$ of $\text{Sp}(V_{\#}, \Omega_{\#})$.

For $U \in \text{Mp}^c(V, \Omega)$ we define $U_{\#} \in \text{Aut } \mathbf{H}_{\#}$ by

$$U_{\#}(T) = UTU^{-1}$$

whenever $T \in \mathbf{H}_{\#}$. If $\sigma(U) = g$ and $x, y \in V$ then

$$\begin{aligned} U_{\#}W_{\#}(x \oplus y)U_{\#}^{-1}T &= UW(x)U^{-1}TUW(y)^{-1}U^{-1} \\ &= W(gx)TW(gy)^{-1} = W_{\#}(g_{\#}(x \oplus y))T \end{aligned}$$

whenever $T \in \mathbf{H}_{\#}$. As a consequence, $U_{\#}$ lies in $\text{Mp}^c(V_{\#}, \Omega_{\#})$ and $\sigma(U_{\#}) = g_{\#} = \phi(g)$.

(2.6) THEOREM. There exists a unique Lie group morphism θ making the diagram

$$\begin{array}{ccc} & & \text{Mp}(V_{\#}, \Omega_{\#}) \\ & \nearrow \theta & \downarrow \sigma \\ \text{Sp}(V, \Omega) & \xrightarrow{\phi} & \text{Sp}(V_{\#}, \Omega_{\#}) \end{array}$$

commute.

PROOF. If $\lambda \in U(1) \subset \text{Mp}^c(V, \Omega)$ then $\lambda_{\#}$ is the identity on $\mathbf{H}_{\#}$; the map

$$\text{Mp}^c(V, \Omega) \rightarrow \text{Mp}^c(V_{\#}, \Omega_{\#}): U \mapsto U_{\#}$$

therefore factors through $\text{Sp}(V, \Omega)$ to define

$$\theta: \text{Sp}(V, \Omega) \rightarrow \text{Mp}^c(V_{\#}, \Omega_{\#}).$$

Since $\mathrm{Sp}(V, \Omega)$ is connected and semisimple it has no nontrivial characters. In particular, if η is the standard unitary character of $\mathrm{Mp}^c(V_\#, \Omega_\#)$ then $\eta \circ \theta$ is trivial; it follows that θ takes its values in $\mathrm{Mp}(V_\#, \Omega_\#)$. This proves existence. Two candidates for θ must differ by a character of $\mathrm{Sp}(V, \Omega)$ and therefore coincide, hence uniqueness. $\square$

If $a, b \in \mathbf{H}$ then $a \otimes \bar{b}$ will denote the rank-one operator on $\mathbf{H}$ defined by

$$(a \otimes \bar{b})(e) = \langle e, b \rangle a$$

for $e \in \mathbf{H}$, where $\langle \cdot, \cdot \rangle$ is the Hilbert structure on $\mathbf{H}$. The linear span of $\{a \otimes \bar{b} \mid a, b \in \mathbf{H}\}$ is dense in $\mathbf{H}_\#$.

Let $\mathcal{E}_\# \subset \mathbf{H}_\# \subset \mathcal{E}'_\#$ be the rigged Hilbert space defined by $W_\#$: thus, $\mathcal{E}_\#$ is the space of smooth vectors for $W_\#$ and $\mathcal{E}'_\#$ its antidual. The linear span of $\{a \otimes \bar{b} \mid a, b \in \mathcal{E}\}$ is dense in the Fréchet space $\mathcal{E}_\#$; elements of $\mathcal{E}'_\#$ are therefore determined by their effect on the vectors $a \otimes \bar{b}$ for $a, b \in \mathcal{E}$.

If $\alpha, \beta \in \mathcal{E}'$ then we define $\alpha \otimes \beta \in \mathcal{E}'_\#$ by

$$(\alpha \otimes \beta)(a \otimes \bar{b}) = \alpha(a) \overline{\beta(b)}$$

for $a, b \in \mathcal{E}$. This defines a sesquilinear map

$$(2.7) \quad \mathcal{E}' \times \mathcal{E}' \rightarrow \mathcal{E}'_\#: (\alpha, \beta) \mapsto \alpha \otimes \beta$$

which will be useful later.

3. Vacuum states for polarizations. Let F be a polarization of (V, Ω) ; view F as an abelian complex Lie algebra. By virtue of the commutation relations (2.2), $\dot{W}^C$ defines a representation of F on $\mathcal{E}'$; in this way, $\mathcal{E}'$ becomes an F -module. The Lie algebra cohomology $H^*(F; \mathcal{E}')$ of F with values in $\mathcal{E}'$ is that of the complex

$$(3.1) \quad \rightarrow \bigwedge^{j-1} F^* \otimes \mathcal{E}' \xrightarrow{d} \bigwedge^j F^* \otimes \mathcal{E}' \xrightarrow{d} \bigwedge^{j+1} F^* \otimes \mathcal{E}' \rightarrow$$

with

$$d\gamma(v_0, \dots, v_j) = \sum_{k=0}^j (-1)^k \dot{W}^C(v_k) \gamma(v_0, \dots, \hat{v}_k, \dots, v_j)$$

for $\gamma \in \bigwedge^j F^* \otimes \mathcal{E}'$; $v_0, \dots, v_j \in F$, where $\hat{}$ denotes omission. This cohomology has been computed by Blattner and Rawnsley, with the following result.

(3.2) THEOREM [2]. $H^j(F; \mathcal{E}')$ is a complex line if $j = q(F)$ and is zero otherwise. $\square$

We refer to the complex line $H^{q(F)}(F; \mathcal{E}')$ as the vacuum state for the polarization F .

(3.3) REMARK. Note that $H^0(F; \mathcal{E}')$ is naturally the space $(\mathcal{E}')^F$ of invariant vectors for F , defined by

$$(\mathcal{E}')^F = \{f \in \mathcal{E}' \mid v \in F \Rightarrow \dot{W}^C(v)f = 0\}.$$

It is a particular consequence of (3.2) that this space vanishes unless F is positive, in which case it is one-dimensional. $\square$

Denote by $\mathrm{Sp}(V, \Omega; F)$ the stabilizer of the polarization F under the action of $\mathrm{Sp}(V, \Omega)$ on $\mathrm{Lag}(V^C, \Omega^C)$. Denote by $\mathrm{Mp}^c(V, \Omega; F)$ the full preimage of $\mathrm{Sp}(V, \Omega; F)$

in $\text{Mp}^c(V, \Omega)$ under σ . Let $U \in \text{Mp}^c(V, \Omega; F)$ with $\sigma(U) = g$. By virtue of (2.4), U acts on the complex (3.1) and so on the cohomology $H^*(F; \mathcal{E}')$; in particular, U acts on the complex line $H^{q(F)}(F; \mathcal{E}')$ as multiplication by a scalar $\tau_F(U)$. Write $\text{Det}_F(U)$ for the complex determinant of g^C on F .

The characters τ_F and Det_F of $\text{Mp}^c(V, \Omega; F)$ are related to the standard unitary character η as follows.

$$(3.4) \text{ THEOREM [2]. } (\tau_F)^2 \cdot \text{Det}_F = \eta. \quad \square$$

Let G be another polarization of (V, Ω) . The canonical isomorphisms

$$\bigwedge^j (F \oplus \overline{G})^* = \bigoplus_{s+t=j} \left(\bigwedge^s F^* \otimes \bigwedge^t \overline{G}^* \right)$$

define sesquilinear maps

$$\bigwedge^s F^* \times \bigwedge^t G^* \rightarrow \bigwedge^{s+t} (F \oplus \overline{G})^*$$

which tensor up with (2.7) to yield sesquilinear maps

$$(3.5) \quad \left(\bigwedge^s F^* \otimes \mathcal{E}' \right) \times \left(\bigwedge^t G^* \otimes \mathcal{E}' \right) \rightarrow \bigwedge^{s+t} (F \oplus \overline{G})^* \otimes \mathcal{E}'_{\#}$$

for all nonnegative integers s, t .

The vacuum states for F, G and $F \oplus \overline{G}$ arise from complexes analogous to (3.1). The complexes for F and G are paired into that for $F \oplus \overline{G}$ via the maps (3.5). These pairings satisfy a Leibniz rule relative to the differentials in the complexes; they therefore induce pairings in cohomology. In view of (3.2) the only nontrivial cohomology pairing is that of the vacuum states. We thus have the following theorem of Künneth type.

(3.6) THEOREM. *There is a canonical nonsingular sesquilinear pairing of vacuum states*

$$H^{q(F)}(F; \mathcal{E}') \times H^{q(G)}(G; \mathcal{E}') \rightarrow H^q(F \oplus \overline{G}; \mathcal{E}'_{\#})$$

where $q = q(F \oplus \overline{G}) = q(F) + q(G)$.

PROOF. We need only demonstrate nonsingularity. Choose a maximal subspace $F_- \subset F$ on which H is negative-definite and let F_+ be the H -orthocomplement of F_- in F ; $F' = \overline{F_-} \oplus F_+$ is then a positive polarization and according to [2] the nonzero vectors in

$$\bigwedge^{q(F)} F_-^* \otimes (\mathcal{E}')^{F'} \subset \bigwedge^{q(F)} F^* \otimes \mathcal{E}'$$

are closed but not exact relative to the complex (3.1). Apply this to both F and G . Choose nonzero vectors $\phi \in \bigwedge^{q(F)} F_-^*$, $\psi \in \bigwedge^{q(G)} G_-^*$, $\alpha \in (\mathcal{E}')^{F'}$, $\beta \in (\mathcal{E}')^{G'}$; $\phi \otimes \overline{\psi}$ and $\alpha \otimes \overline{\beta}$ are then basis vectors for $\bigwedge^q (F_- \oplus \overline{G_-})^*$ and $(\mathcal{E}'_{\#})^{F' \oplus \overline{G'}}$. $\phi \otimes \alpha$ and $\psi \otimes \beta$ are paired to

$$(\phi \otimes \overline{\psi}) \otimes (\alpha \otimes \overline{\beta}) \in \bigwedge^q (F \oplus \overline{G})^* \otimes \mathcal{E}'_{\#};$$

this vector is closed but not exact. $\square$

It is important to determine how $\text{Mp}^c(V, \Omega)$ acts on these vacuum state pairings. Let $U \in \text{Mp}^c(V, \Omega)$ with $\sigma(U) = g$. Via the standard actions of g and $g_\#$ and the (extended) metaplectic actions of U and $U_\#$, U maps the complexes for $F, G, F \oplus \bar{G}$ to those for $gF, gG, gF \oplus g\bar{G} = g_\#(F \oplus \bar{G})$. These operations of $\text{Mp}^c(V, \Omega)$ respect the differentials and pairings of the complexes; the induced natural operations of $\text{Mp}^c(V, \Omega)$ on cohomology respect the natural cohomology pairings. These arguments yield the following result.

(3.7) THEOREM. *The diagram*

$$\begin{array}{ccc} H^s(F; \mathcal{E}') \times H^t(G; \mathcal{E}') & \longrightarrow & H^q(F \oplus \bar{G}; \mathcal{E}'_\#) \\ U \cdot \downarrow & & \downarrow U_\# \\ H^s(gF; \mathcal{E}') \times H^t(gG; \mathcal{E}') & \longrightarrow & H^q(g_\#(F \oplus \bar{G}); \mathcal{E}'_\#) \end{array}$$

commutes when $s = q(F)$, $t = q(G)$, $q = q(F \oplus \bar{G})$, where horizontal arrows denote the natural cohomology pairings and where vertical arrows denote the natural operations of $\text{Mp}^c(V, \Omega)$ on cohomology. $\square$

4. Structures on symplectic vector bundles. Let (E, ω) be a real symplectic vector bundle of rank $2m$ over the manifold X . The symplectic frame bundle of (E, ω) is the principal bundle $\text{Sp}(E, \omega)$ on X whose fiber over $x \in X$ is the set of all linear isomorphisms $b: V \xrightarrow{\sim} E_x$ such that

$$v_1, v_2 \in V \Rightarrow \omega_x(bv_1, bv_2) = \Omega(v_1, v_2)$$

and on which the structure group $\text{Sp}(V, \Omega)$ acts by composition on the right.

An Mp^c structure for (E, ω) is a principal $\text{Mp}^c(V, \Omega)$ bundle P on X together with a principal bundle map $P \rightarrow \text{Sp}(E, \omega)$ equivariant with respect to the morphism $\sigma: \text{Mp}^c(V, \Omega) \rightarrow \text{Sp}(V, \Omega)$. The Mp^c structures P_1 and P_2 for (E, ω) are equivalent iff there exists an isomorphism $P_1 \xrightarrow{\sim} P_2$ of principal $\text{Mp}^c(V, \Omega)$ bundles respecting the projections on $\text{Sp}(E, \omega)$. We denote by $\text{Mp}^c[E, \omega]$ the space of equivalence classes $[P]$ of Mp^c structures P for (E, ω) .

Mp (or metaplectic) structures for (E, ω) are defined analogously, replacing $\text{Mp}^c(V, \Omega)$ by $\text{Mp}(V, \Omega)$. Note that any Mp structure extends naturally to an Mp^c structure via inclusion of structure groups.

At a later point we shall discuss questions of existence and classification for these structures; for the present we consider some consequences of their existence.

The extended metaplectic action of $\text{Mp}^c(V, \Omega)$ on $\mathcal{E}'$ associates to each Mp^c structure P for (E, ω) a bundle $\mathcal{E}'(P)$ of symplectic spinors; this bundle comes equipped with a fiberwise operation $\dot{W}^C$ of the complexification E^C , arising from the projective representation (2.3) of V^C on $\mathcal{E}'$. Since both E^C and $\mathcal{E}'(P)$ are associated to P , if $x \in X$ and $p \in P_x$ then p defines isomorphisms

$$V^C \xrightarrow{\sim} E_x^C, \quad \mathcal{E}' \xrightarrow{\sim} \mathcal{E}'(P)_x,$$

and we set

$$\dot{W}_x^C(pv) = p \circ \dot{W}^C(v) \circ p^{-1}$$

for $v \in V^C$; that $\dot{W}^C$ is well defined follows from (2.4).

A polarization of (E, ω) is a complex subbundle F of E^C such that the fiber F_x is a polarization of the symplectic vector space (E_x, ω_x) for each $x \in X$; we assume

throughout that F is regular in the sense that the type of F_x is independent of $x \in X$. The canonical bundle of F is the complex line bundle

$$K^F = \bigwedge^m (F^0) \subset \bigwedge^m (E^{\mathbb{C}})^*$$

where F^0 is the annihilator of F , given by

$$F_x^0 = \{f \in (E_x^{\mathbb{C}})^* \mid v \in F_x \Rightarrow f(v) = 0\}$$

for $x \in X$.

Let P be an Mp^c structure and F a polarization. Fiberwise, $\mathcal{E}'(P)$ is an F -module via $\tilde{W}^{\mathbb{C}}$; we may therefore form the cohomology bundle $H_P^*(F; \mathcal{E}')$ of F with values in $\mathcal{E}'(P)$. The following derives from (3.2).

(4.1) THEOREM [2]. $H_P^j(F; \mathcal{E}')$ is a complex line bundle if $j = q(F)$ and is zero otherwise. $\square$

(4.2) REMARK. The isomorphism class of $H_P^{q(F)}(F; \mathcal{E}')$ depends only on F and the equivalence class of P ; moreover, when F is positive the dependence on F disappears. See [2] and (5.7). $\square$

Let $P(\eta)$ be the Hermitian line bundle on X associated to P via the standard unitary character η of $\text{Mp}^c(V, \Omega)$. The following comes from (3.4).

(4.3) THEOREM [2]. There is a canonical isomorphism

$$H_P^{q(F)}(F; \mathcal{E}') \otimes H_P^{q(F)}(F; \mathcal{E}') \otimes K^F \xrightarrow{\sim} P(\eta);$$

thus, $H_P^{q(F)}(F; \mathcal{E}') \otimes K^F$ is a canonical square root of $P(\eta) \otimes K^F$. $\square$

We call $H_P^{q(F)}(F; \mathcal{E}') \otimes K^F$ the bundle of half-forms for F defined by P .

(4.4) REMARK. If P is the extension of an Mp structure then $P(\eta)$ is trivial and $H_P^{q(F)}(F; \mathcal{E}') \otimes K^F$ is a square root of K^F ; hence the terminology of half-forms. $\square$

At this point we pause to discuss questions of existence and classification.

(E, ω) always admits Mp^c structures, and $\text{Mp}^c[E, \omega]$ is naturally a principal homogeneous space for $H^2(X; \mathbb{Z})$; furthermore, $\text{Mp}^c[E, \omega]$ has a natural base-point (the neutral class) to which the Mp^c structure P belongs iff $\mathcal{E}'(P)^F = H_P^0(F; \mathcal{E}')$ is trivial for F a positive polarization. See [7 and 8].

(E, ω) admits Mp structures iff the second Stiefel-Whitney class $w_2(E)$ is zero, and $\text{Mp}[E, \omega]$ is naturally a principal homogeneous space for $H^1(X; \mathbb{Z}_2)$ when nonempty; in general, $\text{Mp}[E, \omega]$ has no base-point. See [3 and 4].

We can apply the $\#$ -operation to each fiber of (E, ω) ; this produces a symplectic vector bundle $(E_{\#}, \omega_{\#})$.

The Whitney product formula for $E_{\#} = E \oplus E$ gives

$$w_2(E_{\#}) = w_1(E)^2 = 0$$

since E is orientable. From this it follows that $(E_{\#}, \omega_{\#})$ admits metaplectic structures. In fact, much more is true.

(4.5) THEOREM. $(E_{\#}, \omega_{\#})$ admits a canonical metaplectic structure, Q .

PROOF. As a principal $\text{Mp}(V_{\#}, \Omega_{\#})$ bundle, Q is associated to $\text{Sp}(E, \omega)$ via the canonical map $\theta: \text{Sp}(V, \Omega) \rightarrow \text{Mp}(V_{\#}, \Omega_{\#})$; the commutative diagram in (2.6) shows Q to be naturally an Mp structure for $(E_{\#}, \omega_{\#})$. $\square$

(4.6) REMARK. It is readily verified that the extension of Q to an Mp^c structure is neutral: indeed, if $F_\Delta = E_\Delta^C$ is the diagonal polarization of $(E_\#, \omega_\#)$ then $H_Q^0(F_\Delta; \mathcal{E}'_\#)$ is trivial, being associated to $\text{Sp}(E, \omega)$. $\square$

5. Half-form pairings. Let F and G be polarizations of (E, ω) . The (external) direct sum $F \oplus \overline{G}$ is then a polarization of the symplectic vector bundle $(E_\#, \omega_\#)$. Note that we make no assumptions regarding the nature of the intersection $F \cap \overline{G}$.

Recall from (4.5) that $(E_\#, \omega_\#)$ is naturally provided with a metaplectic structure Q , whose extension to an Mp^c structure we denote also by Q . Using Q we define the half-form bundle

$$H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#) \otimes K^{F \oplus \overline{G}}$$

for $F \oplus \overline{G}$, where

$$q = q(F \oplus \overline{G}) = q(F) + q(G).$$

We stress that this complex line bundle is constructed entirely from the symplectic data $(E, \omega), F, G$.

Let P be an Mp^c structure for (E, ω) . Using P we define the half-form bundles

$$H_P^{q(F)}(F; \mathcal{E}') \otimes K^F, \quad H_P^{q(G)}(G; \mathcal{E}') \otimes K^G$$

for F, G . We shall pair these half-form bundles into that for $F \oplus \overline{G}$ defined by Q .

Let $x \in X$; let $s \in H_P^{q(F)}(F; \mathcal{E}')_x$ and $t \in H_P^{q(G)}(G; \mathcal{E}')_x$. A choice of $p \in P_x$ determines isomorphisms of typical fibers with those of associated bundles and so picks out $p^{-1}s \in H^{q(F)}(p^{-1}F_x; \mathcal{E}')$ and $p^{-1}t \in H^{q(G)}(p^{-1}G_x; \mathcal{E}')$ in the cohomology of the polarizations $p^{-1}F_x$ and $p^{-1}G_x$ of (V, Ω) ; $p^{-1}s$ and $p^{-1}t$ give an element of $H^q(p^{-1}F_x \oplus p^{-1}\overline{G}_x; \mathcal{E}'_\#)$ via the pairing in (3.6); we define $\langle s, t \rangle$ to be the image of this element in $H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#)_x$ under the isomorphism determined by the frame $b \in \text{Sp}(E, \omega)_x$ lying below p , regarding b as an element of Q_x by association; this definition is independent of the choice $p \in P_x$ by virtue of the commutative diagram in (3.7). In this way we obtain a canonical nonsingular sesquilinear pairing of vacuum states, thus:

$$(5.1) \quad H_P^{q(F)}(F; \mathcal{E}') \times H_P^{q(G)}(G; \mathcal{E}') \rightarrow H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#).$$

Since there is a canonical isomorphism

$$(5.2) \quad K^F \otimes \overline{K^G} \xrightarrow{\sim} K^{F \oplus \overline{G}}$$

we deduce the following result.

(5.3) THEOREM. *There is a canonical nonsingular sesquilinear pairing of the half-form bundles $H_P^{q(F)}(F; \mathcal{E}') \otimes K^F$, $H_P^{q(G)}(G; \mathcal{E}') \otimes K^G$ into the half-form bundle $H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#) \otimes K^{F \oplus \overline{G}}$. $\square$*

This is our half-form pairing; it can be reformulated as a canonical isomorphism of complex line bundles

$$(5.4) \quad H_P^{q(F)}(F; \mathcal{E}') \otimes K^F \otimes \overline{H_P^{q(G)}(G; \mathcal{E}') \otimes K^G} \xrightarrow{\sim} H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#) \otimes K^{F \oplus \overline{G}}.$$

We emphasize the fact that the receiving space for this half-form pairing is fashioned from the symplectic data alone.

Note that since Q is metaplectic, $H_Q^q(F \oplus \overline{G}; \mathcal{E}'_\#) \otimes K^{F \oplus \overline{G}}$ is a canonical square root of $K^{F \oplus \overline{G}}$: see (4.4). This is as we should expect of a half-form pairing. As a corollary we recover the following result due to Rawnsley [5].

(5.5) THEOREM. *If F and G are arbitrary polarizations of (E, ω) then $c[K^F] \equiv c[K^G] \pmod{2}$ where $c[\cdot]$ denotes Chern class.*

PROOF. Since $K^F \otimes \overline{K^G}$ is isomorphic to $K^{F \oplus \overline{G}}$ and $K^{F \oplus \overline{G}}$ is a square, it follows that

$$\pmod{2} c[K^F \otimes \overline{K^G}] = 0;$$

but

$$c[K^F \otimes \overline{K^G}] = c[K^F] - c[K^G]. \quad \square$$

We close by remarking on particular cases.

Suppose F and G are positive polarizations; assume F and G to be transverse in the sense that $F \cap \overline{G} = 0$. In this case $H_Q^0(F \oplus \overline{G}; \mathcal{E}'_\#)$, or equivalently $\mathcal{E}'_\#(Q)^{F \oplus \overline{G}}$, is canonically trivial; this can be verified as follows. Let $x \in X$. By association, each $b \in \text{Sp}(E, \omega)_x$ becomes an element of Q_x . The vacuum states $(\mathcal{E}')^{b^{-1}F_x}$ and $(\mathcal{E}')^{b^{-1}G_x}$ for the transverse positive polarizations $b^{-1}F_x$ and $b^{-1}G_x$ of (V, Ω) admit a canonical sesquilinear pairing into $\mathbf{C}$ which extends the inner product on $\mathbf{H}$; this is proved in [3] for real polarizations and in [6, 7, 8] for the general positive case. If $\phi \in (\mathcal{E}')^{b^{-1}F_x}$ and $\psi \in (\mathcal{E}')^{b^{-1}G_x}$ pair to 1 $\in \mathbf{C}$ then $b(\phi \otimes \overline{\psi})$ is a canonical element of $\mathcal{E}'_\#(Q)^{F \oplus \overline{G}}$ defined independently of ϕ, ψ , and $b \in \text{Sp}(E, \omega)_x$.

(5.6) THEOREM. *If (F, G) is a transverse pair of positive polarizations then $H_Q^0(F \oplus \overline{G}; \mathcal{E}'_\#)$ is canonically trivial.* $\square$

(5.7) REMARK. From (5.1) and (5.6) it is clear that if P is an Mp^c structure then the line bundles $H_P^0(F; \mathcal{E}') = \mathcal{E}'(P)^F$ are isomorphic for all positive polarizations F : we may always take G to be a fixed strictly positive polarization. $\square$

Now suppose the pair (F, G) of positive polarizations to be regular in the sense that $F \cap \overline{G}$ has constant rank; then $F \cap \overline{G} = D^C$ for a subbundle $D \subset E$ on which ω is identically zero [1]. In this case $H_Q^0(F \oplus \overline{G}; \mathcal{E}'_\#) = \mathcal{E}'_\#(Q)^{F \oplus \overline{G}}$ is canonically isomorphic to the trivial line bundle $\mathcal{D}^1(D)$ of densities on D ; a proof of this runs as follows. For $x \in X$ denote by $D_x^\perp$ the space of all $v \in E_x$ such that $\omega_x(v, w) = 0$ whenever $w \in D_x$. The quotient bundle $D^\perp/D$ has a natural symplectic structure ω_D ; moreover, $F_D = F/D^C$ and $G_D = G/D^C$ are transverse positive polarizations of $(D^\perp/D, \omega_D)$. It is shown in [6, 7, 8] that when $D \neq D^\perp$, each Mp^c structure P for (E, ω) induces an Mp^c structure P_D for $(D^\perp/D, \omega_D)$ with a canonical isomorphism from the symplectic spinors $\mathcal{E}'(P)^D$ annihilated by D to the bundle $\mathcal{E}'(P_D) \otimes \mathcal{D}^{1/2}(D)$, restricting to an isomorphism $\mathcal{E}'(P)^Z \xrightarrow{\sim} \mathcal{E}'(P_D)^{Z_D} \otimes \mathcal{D}^{1/2}(D)$ whenever Z is a positive polarization of (E, ω) with $D^C \subset Z$. Apply this to both F and G . In view of (5.1) and the fact that the half-densities $\mathcal{D}^{1/2}(D)$ pair naturally into $\mathcal{D}^1(D)$ we obtain a canonical isomorphism

$$\mathcal{E}'_\#(Q)^{F \oplus \overline{G}} \xrightarrow{\sim} \mathcal{E}'_\#(Q^D)^{F_D \oplus \overline{G_D}} \otimes \mathcal{D}^1(D),$$

where Q^D is the canonical metaplectic structure for $((D^\perp/D)_\#, (\omega_D)_\#)$. According to (5.6) the bundle $\mathcal{E}'_\#(Q^D)^{F_D \oplus \overline{G_D}}$ is canonically trivial since $F_D \cap \overline{G_D} = 0$; thus

$\mathcal{E}'_{\#}(Q)^{F \oplus \overline{G}}$ is canonically isomorphic to $\mathcal{D}^1(D)$. This argument fails if $G = F = \overline{F}$, since then $D = D^{\perp}$; in this case, a sesquilinear self-pairing of $\mathcal{E}'(P)^F$ into $\mathcal{D}^1(D)$ comes directly from (3.4) and we may again apply (5.1). In any case, we have justified the following result.

(5.8) THEOREM. *If (F, G) is a regular pair of positive polarizations with $F \cap \overline{G} = D^C$ then $H_Q^0(F \oplus \overline{G}; \mathcal{E}'_{\#})$ is canonically isomorphic to $\mathcal{D}^1(D)$. $\square$*

(5.9) REMARK. With the same hypotheses, $K^{F \oplus \overline{G}} = K^F \otimes \overline{K^G}$ is canonically isomorphic to $\mathcal{D}^{-2}(D)$; see [1, 4]. Our half-form pairing thus takes values in $\mathcal{D}^{-1}(D)$ and accords with the regular pairing obtained previously in [1 and 8]. $\square$

(5.10) REMARK. (5.8) can be established without the auxiliary Mp^c structure P , along the following lines. $\omega_{\#}$ is zero on $D \oplus D \subset E_{\#}$; the quotient $(D \oplus D)^{\perp}/(D \oplus D)$ is canonically isomorphic to $(D^{\perp}/D)_{\#}$ as a symplectic vector bundle. The canonical Mp^c structure Q^D on $(D^{\perp}/D)_{\#}$ agrees with that induced on $(D \oplus D)^{\perp}/(D \oplus D)$ from the canonical Mp^c structure on $E_{\#}$; we thus have a canonical isomorphism

$$\mathcal{E}'_{\#}(Q)^{F \oplus \overline{G}} \xrightarrow{\sim} \mathcal{E}'_{\#}(Q^D)^{F_D \oplus \overline{G_D}} \otimes \mathcal{D}^{1/2}(D \oplus D).$$

$\mathcal{E}'_{\#}(Q^D)^{F_D \oplus \overline{G_D}}$ is canonically trivial by (5.6) while $\mathcal{D}^{1/2}(D \oplus D)$ is canonically isomorphic to $\mathcal{D}^1(D)$; hence (5.8). $\square$

Finally, we note that in general the bundle $H_Q^q(F \oplus \overline{G}; \mathcal{E}'_{\#})$ need not even be trivial: it suffices to take F to be positive, to let $G = \overline{F}$, and to suppose that $2c[K^F]$ is nonzero.

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